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Coherent states of SU(N) groups

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Abstract. An explicit and uniform construction for coherent states (CS) for all the SU(N) groups is given and, on this basis, their properties are investigated. The CS are parametrized by the dots of a coset space, which is, in this particular case, the projective space CP^{N-1} and which plays the role of the phase space in the corresponding classical mechanics. The logarithm of the modulus of the CS overlap, being interpreted as a symmetric in the space, gives the Fubini–Study metric in CP^{N-1}. The classical limit is investigated in terms of operator symbols. $h = P^{-1}$ (where P is the signature of the representation) plays the role of Planck's constant. The classical limit of the so called star commutator of the symbols generates the Poisson bracket in the corresponding phase space. The CS form an overcompleted system in the representation space and, as quantum states possess a minimum uncertainty, they minimize an invariant dispersion of the quadratic Casimir operator.

1. Introduction

As is well known, coherent states (CS) are widely and fruitfully used in different areas of theoretical physics [1–5]. The CS introduced by Schrödinger and Glauber turned out to be orbits of the Heisenberg–Weyl group. This observation, by analogy, allowed some general definition of CS for any Lie group [6, 7] to be formulated as orbits of the group factorized with respect to a stationary subgroup. A connection between the CS and the quantization of classical systems, in particular systems with a curved phase space, was also established [8, 9]. By origin, the CS are quantum states, but, at the same time, they are parametrized by dots of the phase space of a corresponding classical mechanics. This circumstance makes them very convenient for analysing the correspondence between quantum and classical descriptions. All this explains the interest in both the investigation of general problems of CS theory and the construction of CS of concrete groups.

The CS of such important physics groups as SU(N) are of special interest, in particular in connection with the description of spinning and isospinning systems. The CS of the group SU(2) are well known and constructed explicitly. One can point out some of the first references [10–15], where these states were constructed from the basis of the well investigated structure of the SU(2) matrices in the fundamental representation. Another approach to the CS construction of the SU(2) group was used by Berezin [8, 9]. This approach was connected with the use of the representations of the SU(2) group in the space of polynomials of the power not more than a given

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one. As to the CS of the SU(N) groups with arbitrary N, their explicit construction, in the framework of the general definition, by means of a direct action of unitary representation operators on some vectors (for example on highest weights), using only commutation relations between generators, is a complicated problem, where complications essentially grow with the number N (one ought to say that many of the properties of the CS can be derived from the general definition without giving them an explicit form [16], and used, for example, in the derivation of $1/N$ decompositions [17], in path integral construction [18] and so on [19]). Nevertheless, the problem can be solved explicitly if appropriate representations of the SU(N) groups, namely representations in the space of polynomials of a fixed power are chosen. Using such representations, we construct here the CS for all the SU(N) groups in a uniform way and, in particular, as orbits of highest weights, and on the basis of this explicit form we investigate some properties of the CS and the problem of the classical limit. The method used can be considered as a generalization of Berezin's method for the SU(2) group in a gauge-invariant form (with an extended number of variables in the coset space), and Gilmor's method of a wavefunction construction for the system of many identical N-level atoms in an external field, with a linear interaction for the generators of the SU(N) algebra [20].

The representations in the space of polynomials with a fixed power are equivalent to the total symmetric irreducible unitary representations of the SU(N) groups. The stationary subgroups of the highest weights, in the case under consideration, are U($N-1$), so that the CS are parametrized by dots of the coset space SU(N)/U($N-1$), which plays the role of the phase space of the corresponding classical mechanics and, at the same time, is the well known projective space CP^{N-1}. The logarithm of the modulus of the CS overlap, being interpreted as a symmetric in the space CP^{N-1}, generates the Fubini–Study metric in the space. The CS form an overcompleted system in the representation space and, as quantum states, they minimize an invariant dispersion of the quadratic Casimir operator. The classical limit is investigated in terms of operator symbols, which are constructed as mean values of the operators in the CS. The quantity $h = P^{-1}$, where P is the signature of a representation, plays the role of the Planck constant. The classical limit of the so called star commutator of the symbols generates the classical Poisson bracket in the corresponding phase space. In addition, we present a direct way of constructing the SU(2) CS in a Fock space. This derivation of the CS of the SU(2) is technically new and instructive to our mind as it allows both ways to be compared and problems with the $N > 2$ generalization can be better understood. The present work is a continuation of our papers [21], where some of the results were preliminarily expounded.

2. Construction of CS of the SU(N) groups

We are going to construct the CS of the SU(N) groups as orbits in some irreducible representations of the groups, factorized with respect to stationary subgroups. First, we describe the corresponding representations.

Let C^N be the N-dimensional space of complex lines $z = (z_\mu), \mu = \overline{1, N}$ with the scalar product $(z, z')_C = \sum_\mu \bar{z}_\mu z'_\mu, \mu = \overline{1, N}$ and $\widetilde{C}^N$ is the dual space of complex columns with the scalar product $(\bar{z}, \bar{z}')_{\widetilde{C}} = \sum_\mu \bar{z}^\mu \bar{z}^\mu$. The anti-isomorphism is given by the relation $z \leftrightarrow \bar{z} \Leftrightarrow \bar{z}_\mu = \bar{z}^\mu$. The mixed (Dirac) scalar product between the elements of C^N and $\tilde{C}^N$ is defined by the equation

$$\langle z', \tilde{z} \rangle = (\tilde{z}', \tilde{z})_{\tilde{C}} = \overline{(z', z)_C} = z'_\mu \tilde{z}^\mu. \tag{1}$$

Let g be the matrices of the fundamental representation of the SU(N) group. This representation induces irreducible representations of the group in the spaces Π_P and $\tilde{\Pi}_P$ of polynomials of a fixed power P on the vectors z and $\tilde{z}$ respectively,

$$\begin{aligned} &T(g)\Psi_P(z) = \Psi_P(z_g) \qquad z_g = zg \qquad \Psi_P \in \Pi_P \\ &\tilde{T}(g)\Psi_P(\tilde{z}) = \Psi_P(\tilde{z}_g) \qquad \tilde{z}_g = g^{-1}\tilde{z} \qquad \Psi_P(\tilde{z}) \in \tilde{\Pi}_P. \end{aligned} \tag{2}$$

The anti-isomorphism $z \leftrightarrow \tilde{z}$ induces the correspondence $\Psi_P(\tilde{z}) = \overline{\Psi_P(z)}$. The representation (2) is equivalent to the one on total symmetric tensors of signature P. So, we will call P the signature of the irreducible representation.

Clearly the monomials

$$\begin{aligned} &\Psi_{P,\{n\}}(z) = \sqrt{\frac{P!}{n_1! \ldots n_N!}} z_1^{n_1} \ldots z_N^{n_N} \\ &\{n\} = \left\{ n_1, \ldots, n_N \middle| \sum_\mu n_\mu = P \right\} \end{aligned} \tag{3}$$

form a discrete basis in Π_P, and the monomials $\Psi_{P,\{n\}}(\tilde{z}) = \overline{\Psi_{P,\{n\}}(z)}$ form a basis in $\widetilde{\Pi_P}$. The monomials obey the remarkable relation

$$\sum_{\{n\}} \Psi_{P,\{n\}}(z') \Psi_{P,\{n\}}(\tilde{z}) = \langle z', \tilde{z} \rangle^P \tag{4}$$

which is group invariant on account of the invariance of the scalar product (1) under the group transformation, $\langle z'_g, \tilde{z}_g \rangle = \langle z', \tilde{z} \rangle$. We also introduce the scalar product of two polynomials

$$\begin{aligned} &\langle \Psi_P | \Psi'_P \rangle = \int \overline{\Psi_P(z)} \Psi'_P(z) \, \mathrm{d}\mu_P(\tilde{z}, z) \\ &\mathrm{d}\mu_P(\tilde{z}, z) = \frac{(P + N - 1)!}{(2\pi)^N P!} \delta\left(\sum |z_\mu|^2 - 1 \right) \prod \mathrm{d}\tilde{z}_\nu \mathrm{d}z_\nu \\ &\mathrm{d}\tilde{z}\, \mathrm{d}z = \mathrm{d}(|z|^2)\mathrm{d}(\arg z). \end{aligned} \tag{5}$$

Using the integral

$$\int_0^1 \mathrm{d}\rho_1 \ldots \int_0^1 \mathrm{d}\rho_N \delta\left(\sum \rho_\mu - 1 \right) \prod_{\nu=1}^N \rho_\nu^{n_\nu} = \frac{\prod_{\nu=1}^N n_\nu!}{\left(\sum_{\nu=1}^N n_\nu + N - 1 \right)!}$$

it is easy to verify that the orthonormality relation holds:

$$\langle \Psi_{P,\{n\}} | \Psi_{P,\{n'\}} \rangle = \langle P, n | P, n' \rangle = \delta_{\{n\},\{n'\}}. \tag{6}$$

The completeness relation takes place as well

$$\sum_{\{n\}} |P,n\rangle\langle P,n| = I_P \tag{7}$$

where $|P,n\rangle$ and $\langle P,n|$ are Dirac's notation for the vectors $\Psi_{P,\{n\}}(z)$ and $\Psi_{P,\{n\}}(\bar{z})$ respectively, and I_P is the identity operator in the irreducible space of representation of signature P.

It is convenient to introduce the operators $a^\dagger_\mu$ and a^μ, which act on the basis vectors by formulae

$$a^\dagger_\mu \Psi_{P,\{n\}}(z) = z_\mu \Psi_{P,\{n\}}(z) \rightarrow a^\dagger_\mu |P,n\rangle = \sqrt{\frac{n_\mu+1}{P+1}}|P+1,\dots,n_\mu+1,\dots\rangle$$

$$a^\mu \Psi_{P,\{n\}}(z) = \frac{\partial}{\partial z_\mu}\Psi_{P,\{n\}}(z) \rightarrow a^\mu|P,n\rangle = \sqrt{Pn_\mu}|P-1,\dots,n_\mu-1,\dots\rangle$$

$$[a^\mu, a^\dagger_\nu] = \delta^\mu_\nu, [a^\mu, a^\nu] = [a^\dagger_\mu, a^\dagger_\nu] = 0. \tag{8}$$

One can find the action of these operators on the left,

$$\langle P,n|a^\dagger_\mu = \sqrt{\frac{n_\mu}{P}}\langle P-1,\dots,n_\mu-1,\dots| = \frac{1}{P}\frac{\partial}{\partial \bar{z}^\mu}\Psi_{P,\{n\}}(\bar{z})$$

$$\langle P,n|a^\mu = \sqrt{(P+1)(n_\mu+1)}\langle P+1,\dots,n_\mu+1,\dots|$$
$$= (P+1)\bar{z}^\mu\Psi_{P,\{n\}}(\bar{z}). \tag{9}$$

Their quadratic combinations A^ν_μ can serve as generators in each irreducible representation of signature P,

$$A^\nu_\mu = a^\dagger_\mu a^\nu = z_\mu\frac{\partial}{\partial z_\nu} \qquad [A^\nu_\mu, A^\kappa_\lambda] = \delta^\nu_\lambda A^\kappa_\mu - \delta^\kappa_\mu A^\nu_\lambda \tag{10}$$

$$A^\nu_\mu|P,n\rangle = \sqrt{n_\nu(n_\mu+1)}|P,\dots,n_\nu-1,\dots,n_\mu+1,\dots\rangle \qquad \mu\neq\nu$$

$$A^\mu_\mu|P,n\rangle = n_\mu|P,n\rangle \qquad \sum_\mu A^\mu_\mu|P,n\rangle = P|P,n\rangle.$$

Clearly, the A^μ_μ are Cartan's generators and $(n_1,\dots,n_N)$ are the weight vectors. The independent generators $\hat{\Gamma}_a, a = \overline{1, N^2-1}$, can be expressed in terms of the operators A^ν_μ,

$$\hat{\Gamma}_a = (\Gamma_a)^\nu_\mu A^\mu_\nu \qquad \left[\hat{\Gamma}_a, \hat{\Gamma}_b\right] = \mathrm{i}f_{abc}\hat{\Gamma}_c \tag{11}$$

where Γ_a are the generators in the fundamental representation, $[\Gamma_a,\Gamma_b] = \mathrm{i}f_{abc}\Gamma_c$ and f_{abc} are the structure constants of the SU(N) group. The quadratic Casimir operator $C_2 = \sum_a \hat{\Gamma}^2_a$ can only be expressed via the operators A^ν_μ by means of the well known formula

$$\sum_a (\Gamma_a)^\nu_\mu (\Gamma)^\kappa_\lambda = \frac{1}{2}\delta^\nu_\lambda\delta^\kappa_\mu - \frac{1}{2N}\delta^\nu_\mu\delta^\kappa_\lambda \tag{12}$$

and evaluated in every irreducible representation explicitly,

$$C_2 = \frac{1}{2}\tilde{A}^{\nu}_{\mu}\tilde{A}^{\mu}_{\nu} = \frac{P(N+P)(N-1)}{2N} \qquad \tilde{A}^{\nu}_{\mu} - \frac{\delta^{\nu}_{\mu}}{N}\sum_{\lambda} A^{\lambda}_{\lambda}.$$

Now we are going to construct the orbits of highest weights (of a vector of the basis (3) with the maximal length $\sqrt{\sum n_{\mu}^2} = P$). Let this highest weight be the state $\Psi_{P,\{P,0...0\}}(z) = (z_1)^P$. Then we get, in accordance with (2),

$$T(g)\Psi_{P,\{P,0...\}}(z) = [z_{\mu}g_1^{\mu}]^P = \langle z, \tilde{u}\rangle^P \qquad \tilde{u}^{\mu} = g_1^{\mu} \tag{14}$$

where the vector $\tilde{u} \in \tilde{C}^N$ is the first column of the SU(N) matrix in the fundamental representation.

If we interpret the representation space as a Hilbert one in quantum mechanics, then we have to identify all the states which differ from each other by a constant phase. For this let us turn to the states of the orbit (14). One can notice that the transformation $\arg\tilde{u}^{\mu} \to \arg\tilde{u}^{\mu} + \lambda$ changes all the states (14) by the constant phase $\exp(\mathrm{i}P\lambda)$. So, we can treat the transformation as a gauge one in a certain sense. To select only physically different quantum CS from all the states of the orbit, we have to impose a gauge condition on $\tilde{u}$, which fixes the total phase of the orbit (14). Such a condition may be chosen in the form $\sum_{\mu}\arg\tilde{u}^{\mu} = 0$. Taking into account that the quantities $\tilde{u}$ obey the condition $\sum|\tilde{u}^{\mu}|^2 = 1$, by their origin as elements of the first column of the SU(N) matrix, we get the explicit form of the CS of the SU(N) group in the space Π_P:

$$\Psi_{P,\tilde{u}}(z) = \langle z, \tilde{u}\rangle^P \tag{15}$$

$$\sum_{\mu}|\tilde{u}^{\mu}|^2 = 1 \qquad \sum_{\mu}\arg\tilde{u}^{\mu} = 0. \tag{16}$$

In the same way we construct the orbit of the highest weight $\Psi_{P,\{P,0...0\}}(\tilde{z}) = (\tilde{z}^1)^P$ in the space $\tilde{\Pi}_P$, and the corresponding CS have the form

$$\Psi_{P,u}(\tilde{z}) = \langle u, \tilde{z}\rangle^P \tag{17}$$

$$\sum_{\mu}|u_{\mu}|^2 = 1 \qquad \sum_{\mu}\arg u_{\mu} = 0. \tag{18}$$

Clearly, $\Psi_{P,\tilde{u}}(z) = \overline{\Psi_{P,u}(\tilde{z})}, z \leftrightarrow \tilde{z}, u \leftrightarrow \tilde{u}$.

It is easy to see that all the elements of the discrete basis (3) with weight vectors of the form $(n_{\mu} = \delta^{\nu}_{\mu}P, \mu = \overline{1,N})$ belong to the CS set (15) with parameters $(\tilde{u}^{\mu} = \delta^{\mu}_{\nu}P, \mu = \overline{1,N})$. An analogous statement is valid for the dual basis and to the CS (17).

The quantities $\tilde{u}$ and u, which parametrized the CS (15) and (16), are elements of the coset space SU$(N)/$U$(N-1)$, in accordance with the fact that the stationary subgroups of both the initial vectors from the spaces Π_P and $\tilde{\Pi}_P$ are U$(N-1)$. At the same time, the coset space is the so called projective space CP^{N-1} (we remember that the complex projective space is defined as the set of all non-zero vectors z in C^N, where z and $\lambda z, \lambda \neq 0$, are equivalent [22]). Equations (16) or (18) are just the possible conditions which define the projective space. The coordinates u or $\bar{u}$ are called homogeneous in CP^{N-1}. Thus, the constructed CS are parametrized by the elements of the projective space CP^{N-1}, which is a symplectic manifold [22] and can therefore be considered to be the phase space of classical mechanics.

To decompose the CS in the discrete bases, we can use the scalar product (5) directly, but there exists a simpler way. One can use relation (4), as the right-hand side of equation (4) can be treated as CS (15) or (17). Thus, it follows from (4)

$$\Psi_{P,\bar{u}}(z) = \sum_{\{n\}} \Psi_{P,\{n\}}(\bar{u})\Psi_{P,\{n\}}(z). \tag{19}$$

This also implies

$$\langle P,u|P,n\rangle = \Psi_{P,\{n\}}(u) \qquad \langle P,n|P,u\rangle = \Psi_{P,\{n\}}(\bar{u}) \tag{20}$$

where $|P,u\rangle$ and $\langle P,u|$ are Dirac's notation for the CS $\Psi_{P,\bar{u}}(z)$ and $\Psi_{P,u}(\bar{z})$ respectively. So, we come to the statement which is important for understanding the result: the discrete bases in the spaces Π_P and $\bar{\Pi}_P$ are the same as the ones in the CS representation.

The completeness relation for the CS can be extracted from the equation (6). Using formulae (20) in integral (6), we get

$$\int \langle P,n|P,u\rangle\langle P,u|P,n'\rangle \, \mathrm{d}\mu_P(\bar{u},u) = \delta_{\{n\},\{n'\}}.$$

This proves the completeness relation

$$\int |P,u\rangle\langle P,u| \, \mathrm{d}\mu_P(\bar{u},u) = I_P. \tag{21}$$

We have to note that the explicit and uniform construction of the CS for all the groups SU(N) proved to be possible due to the choice of irreducible representations of the groups in the spaces of polynomials of a fixed power. One can try to construct orbits directly, acting, by means of the unitary representation operators, on the highest weights, but an explicit result can be found relatively easy only for the SU(2) group. Later we demonstrate such a way for the latter case in a Fock space.

Let us have a Fock space set up by means of two types of Bose annihilation and creation operators $a_\lambda, a_\lambda^+, \lambda = 1,2, [a_\lambda, a_{\lambda'}^+] = \delta_{\lambda,\lambda'}$. One can realize the commutation relations $[\hat{J}_i, \hat{J}_j] = \mathrm{i}\epsilon_{ijk}\hat{J}_k$ for the generators $\hat{J}_i, i = 1,2,3$, of the SU(2) group in the Fock space, choosing the operators $\hat{J}_i$ in the form (the so called Jordan–Schwinger representation [23])

$$\hat{J}_i = \frac{1}{2}\sum_{\lambda,\lambda'} a_\lambda^+(\sigma_i)_{\lambda'}^{\lambda} a_{\lambda'}$$

where σ_i are Pauli matrices. So one gets

$$\hat{J}_+ = \hat{J}_1 + \mathrm{i}\hat{J}_2 = a_1^+ a_2 \qquad \hat{J}_- = \hat{J}_1 - \mathrm{i}\hat{J}_2 = a_2^+ a_1$$
$$\hat{J}_3 = \tfrac{1}{2}(\hat{n}_1 - \hat{n}_2)$$
$$\hat{J}^2 = \mathcal{J}(\mathcal{J}+1) \qquad \hat{n}_\lambda = a_\lambda^+ a_\lambda \qquad \mathcal{J} = \tfrac{1}{2}(\hat{n}_1 + \hat{n}_2).$$

The occupation numbers basis $|n_1 n_2\rangle = (n_1! n_2!)^{-1/2}(a_1^+)^{n_1}(a_2^+)^{n_2}|0\rangle, n_\lambda = 0, 1, \ldots,$ is correct for the operators $\hat{J}_3$ and $\mathcal{J}$,

$$\mathcal{J}|n_1 n_2\rangle = j|n_1 n_2\rangle \qquad j = \tfrac{1}{2}(n_1 + n_2) = 0, 1/2, 1, 3/2, \ldots$$
$$\hat{J}_3|n_1 n_2\rangle = m|n_1 n_2\rangle \qquad m = \tfrac{1}{2}(n_1 - n_2) = -j, -j+1, \ldots, j-1, j.$$

Thus, we get the well known domains of variations in the eigenvalues j and m. In terms of these quantities the basis $|n_1 n_2\rangle$ can be written in the form $|n_1 n_2\rangle = |j, m\rangle = ((j+m)!(j-m)!)^{-1/2}(a_1^+)^{j+m}(a_2^+)^{j-m}|0\rangle$. So

$$\mathcal{J}|j, m\rangle = j|j, m\rangle \qquad \hat{J}_3|j, m\rangle = m|j, m\rangle$$
$$\hat{J}_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$
$$\hat{J}_\pm|j, \pm j\rangle = 0 \qquad |j, j\rangle = \frac{(a_1^+)^{2j}}{(2j)!}|0\rangle \qquad |j, -j\rangle = \frac{(a_2^+)^{2j}}{(2j)!}|0\rangle. \quad (22)$$

It is known that an element g of the SU(2) group can be parametrized by the Euler angles $\varphi_1, \theta, \varphi_2$, and presented in the form

$$g = \begin{pmatrix} \cos\frac{1}{2}\theta e^{i(\varphi_1+\varphi_2)/2} & i\sin\frac{1}{2}\theta e^{i(\varphi_1-\varphi_2)/2} \\ i\sin\frac{1}{2}\theta e^{-i(\varphi_1-\varphi_2)/2} & \cos\frac{1}{2}\theta e^{-i(\varphi_1+\varphi_2)/2} \end{pmatrix} = e^{i\varphi_1 J_3} e^{i\theta J_1} e^{i\varphi_2 J_3} \quad (23)$$

where $J_i = \sigma_i/2$ are the generators of the SU(2) group in the fundamental representation. Therefore, the operators of the representation $T(g)$, acting in the Fock space, can be written as

$$T(g) = e^{i\varphi_1 \hat{J}_3} e^{i\theta \hat{J}_1} e^{i\varphi_2 \hat{J}_3}.$$

One can derive the formula

$$\exp(\alpha\hat{J}_+ - \alpha^* \hat{J}_-) = \exp\left(-\frac{|\alpha|}{\alpha}\tan|\alpha|\hat{J}_-\right)\exp(2\ln|\cos|\alpha||\hat{J}_3)\exp\left(\frac{\alpha}{|\alpha|}\tan|\alpha|\hat{J}_+\right) \quad (24)$$

which allows one to present the operators $T(g)$ in a more appropriate form for further calculations (an analogue of the normal form),

$$T(g) = e^{i\varphi_1 \hat{J}_3} e^{i\tan\frac{1}{2}\theta \hat{J}_-} (\cos\tfrac{1}{2}\theta)^{2\hat{J}_3} e^{i\tan\frac{1}{2}\theta \hat{J}_+} e^{i\varphi_2 \hat{J}_3}. \quad (25)$$

It is clear that the linear envelope of vectors from the Fock space, with j fixed, forms an invariant and irreducible, relative to the representation $T(g)$, subspace of dimensionality $2j + 1$. Let us construct the orbit of the highest weight $|j, j\rangle$. Denote such an orbit as

$$|j, \varphi_1, \theta, \varphi_2\rangle = T(g)|j, j\rangle. \quad (26)$$

Using equation (25), one can calculate the orbit (26) in the ocupation numbers representation,

$$\langle n_1 n_2 | j, \varphi_1, \theta, \varphi_2\rangle = \delta_{n_1+n_2,2j}\sqrt{\frac{2j!}{n_1!n_2!}}(\cos\tfrac{1}{2}\theta\, e^{i(\varphi_1+\varphi_2)/2})^{n_1}(i\sin\tfrac{1}{2}\theta\, e^{-i(\varphi_1-\varphi_2)/2})^{n_2}$$

$$= \delta_{n_1+n_2,2j}\sqrt{\frac{2j!}{n_1!n_2!}}(\bar{u}^1)^{n_1}(\bar{u}^2)^{n_2} \tag{27}$$

where $\bar{u}^\lambda = g_1^\lambda$ are elements of the first column of the SU(2) matrix (23). If we interpret vectors of the orbit as quantum mechanical states, then all the vectors, which differ from each other by a constant phase factor, have to be identified. In the case under consideration a variation in the Euler angle φ_2 changes only the phase of the orbit (26) (it gives the the factor $\exp i\delta\varphi_2(n_1+n_2)/2 = \exp i\delta\varphi_2 j$). To choose only one representative from the physical equivalent set of vectors we have to impose a gauge condition, which fixes the angle φ_2. This means we have to change to the coset space SU(2)/U(1) where U(1) is the Abelian stationary subgroup of Euler angle φ_2 rotations. The gauge condition we choose is $\varphi_2 = -\frac{1}{2}\pi$, which corresponds to the condition $\sum_\lambda \arg \bar{u}^\lambda = 0$ for the elements of the first column of the SU(2) matrix. Besides, these elements originally obey the condition $\sum_\lambda |\bar{u}^\lambda|^2 = 1$. Thus, physically different elements of the orbit (26), which we call the CS of the SU(2) group, are parametrized by the elements of the projective space CP^1, the latter being the ordinary two-dimensional sphere. In the occupation number representation (so called Bloch states [10]) they have the form

$$\langle n_1 n_2 | j\theta\varphi\rangle = \sqrt{\frac{(2j)!}{n_1!n_2!}}(\bar{u}^1)^{n_1}(\bar{u}^2)^{n_2} \tag{28}$$

$$n_1 + n_2 = 2j \qquad \bar{u}^1 = \cos\tfrac{1}{2}\theta e^{-i\varphi/2} \qquad \bar{u}^2 = \sin\tfrac{1}{2}\theta e^{i\varphi/2}$$

where the angle $\varphi = \frac{1}{2}\pi - \varphi_1$ together with θ and j are the spherical coordinates of the mean values of the isospin vector in the CS, $\langle j\theta\varphi|\hat{J}|j\theta\varphi\rangle = j(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Thus, the CS are parametrized by the dots of the two-dimensional sphere of the quantized radius $P/2 = j = 0, 1/2, 1, \ldots$. The completeness relation holds:

$$\int |j\theta\varphi\rangle\langle j\theta\varphi| d\mu_j(\theta\varphi) = I_P \qquad d\mu_j(\theta\varphi) = \frac{2j+1}{4\pi}\sin\theta\, d\theta\, d\varphi \tag{29}$$

where I_P is the identical operator in the representation space of signature P.

Thus, one can see that the 'direct' way gives the same result for the case of the SU(2) group (compare (28) with (20) and (3)), as we have previously obtained in a simpler way. Moreover, the disentanglement of the representation operators for $N > 2$, similar to formula (25), is a difficult problem, where the complexity essentially grows with the number N. Some results of such a disentanglement for the SU(3) group are presented in [24] and can serve as a illustration of the latter statement.

3. Uncertainty relation

The orbits of each vector of the discrete basis $|P,n\rangle$ (3) and, in particular, the constructed CS are eigenstates for a nonlinear operator C_2', which is defined by its action on an arbitrary vector $|\Psi\rangle$ as

$$C_2'|\Psi\rangle = \sum_a \langle\Psi|\hat{\Gamma}_a|\Psi\rangle\hat{\Gamma}_a|\Psi\rangle. \tag{30}$$

First, we note that $T^{-1}(g)C_2'T(g) = C_2'$, where $T(g)$ are representation operators. Indeed, it follows from the relation $T^{-1}(g)\hat{\Gamma}_a T(g) = t_a^c\hat{\Gamma}_c$ and $[C_2, T(g)] = 0$, that t_a^c is an orthogonal matrix, so that

$$\begin{aligned} T^{-1}(g)C_2'T(g)|\Psi\rangle &= \sum_a \langle\Psi|T^{-1}(g)\hat{\Gamma}_a T(g)|\Psi\rangle T^{-1}(g)\hat{\Gamma}_a T(g)|\Psi\rangle \\ &= \sum_a \langle\Psi|\hat{\Gamma}_a|\Psi\rangle\hat{\Gamma}_a|\Psi\rangle = C_2'|\Psi\rangle. \end{aligned}$$

After that, it is easy to show that the orbit $T(g)|P,n\rangle$ is an eigenstate for C_2'. We write

$$C_2'T(g)|P,n\rangle = T(g)C_2'|P,n\rangle = T(g)\sum_a \langle P,n|\hat{\Gamma}_a|P,n\rangle\hat{\Gamma}_a|P,n\rangle \tag{31}$$

and use formulae (11) and (12) in the right-hand side of (31),

$$\begin{aligned} &\sum_a \langle P,n|\hat{\Gamma}_a|P,n\rangle\hat{\Gamma}_a|P,n\rangle \\ &\qquad = \frac{1}{2}\left[\langle P,n|A_\mu^\nu|P,n\rangle A_\nu^\mu - \frac{1}{N}\sum_\mu A_\mu^\mu\right]|P,n\rangle = \lambda(P,n)|P,n\rangle \end{aligned}$$

$$\lambda(P,n) = \frac{1}{2}\left(\sum_\mu n_\mu^2 - P^2/N\right) = \frac{1}{2}\sum_\mu (n_\mu - P/N)^2.$$

The latter results in

$$C_2'T(g)|P,n\rangle = \lambda(P,n)T(g)|P,n\rangle. \tag{32}$$

The eigenvalue $\lambda(P,n)$ attains the maximum for the highest weights, for which $\sum_\mu n_\mu^2 = P^2 = \max$. The CS $|P,u\rangle$ belong to the orbit of the highest weight $\{n\} = \{P,0,\dots,0\}$. Thus, we get

$$C_2'|P,u\rangle = \frac{P^2(N-1)}{2N}|P,u\rangle. \tag{33}$$

One can introduce a dispersion of the square of the length of the isospin vector [12],

$$\Delta C_2 = \left\langle\Psi\left|\sum_a \hat{\Gamma}_a^2\right|\Psi\right\rangle - \sum_a \langle\Psi|\hat{\Gamma}_a|\Psi\rangle^2 = \langle\Psi|C_2 - C_2'|\Psi\rangle. \tag{34}$$

The dispersion serves as a measure of the uncertainty of the state $|\Psi\rangle$. Due to the properties of the operators C_2 and C_2', it is group invariant and has the least value $P(N-1)/2$ for the orbits of highest weights, particularly for the CS constructed with respect to all the orbits of the discrete basis (3). The relative dispersion of the square of the length of the isospin vector has the value

$$\Delta C_2/C_2 = N/(N+P) \tag{35}$$

in the CS, and tends to zero with $h \to 0, h = 1/P$. This fact already indicates that here the quantity h plays the role of Planck's constant. In section 5 this analogy is traced in more detail.

4. The CS overlap

The CS overlap can be evaluated in different ways. For instance, using the completeness relation (20) and formulae (19), (4), we get

$$\begin{aligned}\langle P,u|P,v\rangle &= \sum_{\{n\}}\langle P,u|P,n\rangle\langle P,n|P,v \\ &= \sum_{\{n\}}\Psi_{P,\{n\}}(u)\Psi_{P,\{n\}}(\bar{v}) = \langle u,\bar{v}\rangle^P.\end{aligned} \tag{36}$$

Comparing the result with equation (14), one can write

$$\langle P,u|P,v\rangle = \Psi_{P,\bar{v}}(u) \tag{37}$$

which once again confirms that the spaces Π_P and $\bar{\Pi}_P$, in quantum mechanical sense, are merely spaces of abstract vectors in the CS representation.

Let $\Psi_P(u)$ be a vector $|\Psi\rangle$ in the CS representation, $\Psi_P(u) = \langle P,u|\Psi\rangle$.

Then the following formula holds

$$\Psi_P(u) = \int \langle P,u|P,v\rangle\Psi_P(v)\,\mathrm{d}\mu_P(\bar{v},v). \tag{38}$$

This means the CS overlap plays the role of the δ-function in the CS representation.

The modulus of the CS overlap (36) possesses the following properties:

$$\begin{aligned}&|\langle P,u|P,v\rangle| < 1 \qquad \lim_{P\to\infty}|\langle P,u|P,v\rangle = 0 \qquad \text{if } u \neq v \\ &|\langle P,u|P,v\rangle| = 1 \qquad \text{only if } u = v.\end{aligned} \tag{39}$$

This follows from the Cauchy inequality for the scalar product (1), $|\langle u,\bar{v}\rangle| \leqslant \sqrt{\langle u,\bar{u}\rangle\langle v,\bar{v}\rangle}$, and from the conditions on the parameters of the CS, $\langle u,\bar{u}\rangle = \langle v,\bar{v}\rangle = 1$.

We can introduce a function $s(u,v)$ of the coordinates of two points of the projective space CP^{N-1},

$$s^2(u,v) = -\ln|\langle P,u|P,v\rangle|^2 = -P\ln|\langle u,\bar{v}\rangle|^2. \tag{40}$$

The properties of the modulus of the CS overlap (39) allow us to interpret the function as symmetric. We remember that a real and positive symmetric obeys only two axioms of a distance ($s(u,v) = s(v,u)$ and $s(u,v) = 0$, if and only if $u = v$), except the triangle axiom. For the CS of the Heisenberg–Weyl group the function $s^2(u,v) = -\ln|\langle u|v\rangle|^2 = |u-v|^2$, and gives, in fact, the square of the distance on the complex plane of the CS parameters. It turns out that, in the case under consideration, the symmetric $s(u,v)$ generates the metric in the projective space CP^{N-1}. To demonstrate this, it is convenient to change from the homogeneous coordinates u_μ, subjected to the supplemental conditions (18), to the local independent coordinates in CP^{N-1}. For instance, in the domain where $u_N \neq 0$, we introduce the local coordinates $\alpha_i, i = \overline{1, N-1}$,

$$\alpha_i + u_i/u_N \tag{41}$$

$$u_i = \alpha_i u_N \qquad u_N = \frac{\exp(-(\mathrm{i}/N)\sum \arg \alpha_k)}{\sqrt{1+\sum|\alpha_k|^2}}.$$

In local coordinates (41) the symmetric (40) takes the form

$$s^2(\alpha,\beta) = -P\ln\frac{\lambda(\alpha,\bar{\beta})\lambda(\beta,\bar{\alpha})}{\lambda(\alpha,\bar{\alpha})\lambda(\beta,\bar{\beta})} \tag{42}$$

where $\lambda(\alpha,\bar{\beta}) = 1 + \sum_i \alpha_i\bar{\beta}_i$.

Thus, we are in position to calculate the square of the 'distance' between two infinitesimal close points α and $\alpha + \mathrm{d}\alpha$. For the $\mathrm{d}s^2$, which is defined as the quadratic part of the decomposition of $s^2(\alpha, \alpha + \mathrm{d}\alpha)$ in the powers of $\mathrm{d}\alpha$, we find

$$\mathrm{d}s^2 = g_{i\bar{k}}\mathrm{d}\alpha_i\mathrm{d}\bar{\alpha}_k \qquad g_{i\bar{k}} = P\lambda^{-2}(\alpha,\bar{\alpha})\left[\lambda(\alpha,\bar{\alpha})\delta_{ik} - \bar{\alpha}_i\alpha_k\right]$$

$$g_{i\bar{k}} = \frac{\partial^2 F}{\partial\alpha_i\partial\bar{\alpha}_k} \qquad F = P\ln\lambda(\alpha,\bar{\alpha}) \tag{43}$$

$$\det\|g_{i\bar{k}}\| = P^{N-1}\lambda^{-N}(\alpha,\bar{\alpha}) \qquad g^{ki} = \frac{1}{P}\lambda(\alpha,\bar{\alpha})(\delta_{ki} + \bar{\alpha}_k\alpha_i).$$

Now we can recognize in expression (43) the so called Fubini–Study metric of the complex projective space CP^{N-1} with a constant holomorphic sectional curvature $C = 4/P$ [22]. It follows from (43) that we are dealing with a Kahlerian manifold. As is well known, a Kahlerian manifold is a symplectic one and a classical mechanics exists on it. The Poisson bracket has the form:

$$\{f,g\} = \mathrm{i}g^{ki}\left(\frac{\partial f}{\partial\alpha_i}\frac{\partial g}{\partial\bar{\alpha}_k} - \frac{\partial f}{\partial\bar{\alpha}_k}\frac{\partial g}{\partial\alpha_i}\right). \tag{44}$$

In the next section we show that the classical limit of the star commutator of the operators symbols in the CS generates this Poisson bracket.

5. The classical limit

One of the advantages of CS is that they allow operator symbols to be constructed in a simple way, i.e. a correspondence between operators and classical functions on the phase space of a system. The reproduction of manipulations with operators in the symbols language is equivalent to the quantization problem. This approach to the quantization was developed by Berezin [8, 25]. In this section we investigate the classical limit in terms of operator symbols constructed by means of the CS.

Let us turn to the so called covariant symbol [25], which is, in fact, the mean value of an operator $\hat{A}$ in the CS,

$$Q_A(u,\bar{u}) = \langle P,u|\hat{A}|P,u\rangle. \tag{45}$$

We also restrict ourselves to operators which are some polynomial functions on the generators, of power not more than some given $M < P$. Such operators can be written via the operators $a^\dagger_\mu, a^\nu$, using (10), (11), and be presented in the 'normal' form,

$$\hat{A} = \sum_{K=0}^{M} A^{\mu_1\ldots\mu_K}_{\nu_1\ldots\nu_K}\, a^\dagger_{\mu_1}\ldots a^\dagger_{\mu_K} a^{\nu_1}\ldots a^{\nu_K}. \tag{46}$$

It is easy to find the action of the operators $a^\dagger_\mu, a^\nu$ on the CS and to calculate the symbols (45),

$$a^\dagger_\mu|P,u\rangle = \frac{1}{P+1}\frac{\partial}{\partial\bar{u}^\mu}|P+1,u\rangle \qquad a^\mu|P,u\rangle = P\bar{u}^\mu|P-1,u\rangle$$
$$\langle P,u|a^\dagger_\mu = u_\mu\langle P-1,u| \qquad \langle P,u|a^\mu = \frac{\partial}{\partial u_\mu}\langle P+1,u|. \tag{47}$$

Thus,

$$Q_A(u,\bar{u}) = \sum_{K=0}^{M} \frac{P!}{(P-K)!} A^{\mu_1\ldots\mu_K}_{\nu_1\ldots\nu_K}\, u_{\mu_1}\ldots u_{\mu_K}\bar{u}_{\nu_1}\ldots\bar{u}_{\nu_K}. \tag{48}$$

Clearly, there is a one-to-one correspondence between an operator and its covariant symbol. In local independent variables α, which were defined in (41), the covariant symbols have the form

$$Q_A(\alpha,\bar{\alpha}) = \sum_{K=0}^{M} \frac{P!}{(P-K)!}\left(1+\sum_i^{N-1}|\alpha_i|^2\right)^{-K} A^{\mu_1\ldots\mu_K}_{\nu_1\ldots\nu_K}\,\alpha_{\mu_1}\ldots\alpha_{\mu_K}\bar{\alpha}_{\nu_1}\ldots\bar{\alpha}_{\nu_K} \tag{49}$$

where the summation over the Greek indices runs from 1 to N as before, but one has to count $\alpha_N = 1$.

In manipulations it is convenient to deal with the symbols

$$\begin{aligned} Q_A(u,\bar{v}) &= \frac{\langle P,u|\hat{A}|P,v\rangle}{\langle P,u|P,v\rangle} \\ &= \sum_{K=0}^{M}\frac{P!}{(P-K)!}\left(\sum_\lambda^N u_\lambda\bar{v}_\lambda\right)^{-K} A^{\mu_1\ldots\mu_K}_{\nu_1\ldots\nu_K}\, u_{\mu_1}\ldots u_{\mu_K}\bar{v}_{\nu_1}\ldots\bar{v}_{\nu_K} \end{aligned} \tag{50}$$

$$Q_A(\alpha,\bar{\beta}) = \sum_{K=0}^{M}\frac{P!}{(P-K)!}\left(1+\sum_i^{N-1}\alpha_i\bar{\beta}_i\right)^{-K}$$
$$\times\, A^{\mu_1\ldots\mu_K}_{\nu_1\ldots\nu_K}\,\alpha_{\mu_1}\ldots\alpha_{\mu_K}\bar{\beta}_{\nu_1}\ldots\bar{\beta}_{\nu_K} \qquad \alpha_N = \beta_N = 1.$$

The symbols $Q_A(\alpha,\bar{\beta})$ are analytical functions on α and $\bar{\beta}$ separately and coincide with the covariant symbols (49) at $\beta \to \alpha$. These symbols but not $\langle P,\alpha|A|P,\beta\rangle$ are a non-diagonal analytical continuation of the covariant symbols.

Using the completeness relation and equation (40), one can find for the symbol of the product of two operators $\hat{A}_1$ and $\hat{A}_2$:

$$Q_{A_1A_2}(u,\bar{u}) = \int Q_{A_1}(u,\bar{v})Q_{A_2}(v,\bar{u})e^{-s^2(u,v)}\,d\mu_P(\bar{v},v). \tag{51}$$

Because $s^2(u,v)$ tends to infinity with $P \to \infty$, if $u \neq v$, and equals zero, if $u = v$, one can conclude that, in that limit, the domain $v \approx u$ gives only a contribution to the integral. Thus,

$$\begin{aligned}\lim_{h\to 0} Q_{A_1A_2}(u,\bar{u}) &= Q_{A_1}(u,\bar{u})Q_{A_2}(u,\bar{u}) \int e^{-s^2(u,v)}\,d\mu_P(\bar{v},v)\\ &= Q_{A_1}(u,\bar{u})Q_{A_2}(u,\bar{u}) \qquad h = 1/P.\end{aligned} \tag{52}$$

The integral in (52) equals unity because of definition (40) and the completeness relation.

If we define the so called star multiplication of symbols according to Beresin [8, 25],

$$Q_{A_1} \star Q_{A_2} = Q_{A_1A_2} \tag{53}$$

then from (52) we have for the covariant symbols:

$$\lim_{h\to 0} Q_{A_1} \star Q_{A_2} = Q_{A_1}Q_{A_2}. \tag{54}$$

Now we are going to obtain the next term in the decomposition of the star multiplication (53) in powers of h. It is is more appropriate to do this in local independent coordinates α, because the decomposition includes differentiation with respect to coordinates. Formula (51) in local coordinates (41) takes the form

$$Q_{A_1A_2}(\alpha,\bar{\alpha}) = \int Q_{A_1}(\alpha,\bar{\beta})Q_{A_2}(\beta,\bar{\alpha})e^{-s^2(\alpha,\beta)}d\mu_P(\bar{\beta},\beta) \tag{55}$$

$$d\mu_P(\bar{\beta},\beta) = \frac{(P+N-1)!}{P!P^{N-1}} \det\|g_{l\bar{m}}(\beta,\bar{\beta})\| \prod_{i=1}^{N-1} \frac{d\,\mathrm{Re}\,\beta_i d\,\mathrm{Im}\,\beta_i}{\pi}$$

where $d\mu_P(\bar{\beta},\beta)$ is proportional to the well known G-invariant measure on CP^{N-1} (see equation (43)). Decomposing the integrand near the point $\beta = \alpha$, and going to the integration over $z = \beta - \alpha$, we get in the zero and first order in power h:

$$\begin{aligned}Q_{A_1A_2}(\alpha,\bar{\alpha}) &= Q_{A_1}(\alpha,\bar{\alpha})Q_{A_2}(\alpha,\bar{\alpha}) + \frac{\partial Q_{A_1}(\alpha,\bar{\alpha})}{\partial\bar{\alpha}_k}\frac{\partial Q_{A_2}(\alpha,\bar{\alpha})}{\partial\alpha_i}\\ &\times \det\|g_{l,\bar{m}}(\alpha,\bar{\alpha})\| \int \bar{z}_k z_i e^{-g_{a\bar{b}}z_a\bar{z}_b} \prod_{j=1}^{N-1} \frac{d\,\mathrm{Re}\,z_j d\,\mathrm{Im}\,z_j}{\pi}\\ &= Q_{A_1}(\alpha,\bar{\alpha})Q_{A_2}(\alpha,\bar{\alpha}) + g^{i\bar{k}}\frac{\partial Q_{A_1}(\alpha,\bar{\alpha})}{\partial\bar{\alpha}_k}\frac{\partial Q_{A_2}(\alpha,\bar{\alpha})}{\partial\alpha_i}\end{aligned} \tag{56}$$

where the matrix g^{ik} was defined in (43) and is proportional to h. Taking into account the expression (44) for the Poisson bracket in the projective space CP^{N-1}, we get for the star commutator of the symbols

$$Q_{A_1} \star Q_{A_2} - Q_{A_2} \star Q_{A_1} = \mathrm{i}\{Q_{A_1}, Q_{A_2}\} + \mathrm{o}(h). \tag{57}$$

Equations (54) and (57) are Berezin's conditions for the classical limit in terms of operator symbols, where the quantity h plays the role of the Planck constant. This property of h has already been noted in section 3, while investigating the uncertainty relation. From this it is easy to see that the length of the isospin vector is proportional to the signature P of a representation. Thus, the classical limit in this case is connected to large values of the isospin vector. In contrast to the ordinary case of the Heisenberg–Weyl group, where the Planck constant is fixed, in the case under consideration the 'Planck constant' can in fact take on different values, which are, however, quantized as of the quantity P is discrete.

6. Conclusion

In the conclusion we wish to stress once again that the explicit construction of the CS for all the SU(N) groups appears to be possible due to an appropriate choice for the irreducible representation of the group in the space of polynomials of a fixed power. The direct way, using the commutation relations of the generators or the structure of the matrices of the group in the fundamental representation, which is possible in the case of the SU(2) group, turns out to be difficult for generalization to any SU(N) group. The explicit form of the SU(N) CS is convenient for the derivation of their properties, for the construction of the operator symbols and for the investigation of the problem of the classical limit.

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References

[1] Klauder J R and Sudarshan E C 1968 *Fundamentals of Quantum Optics* (New York: Benjamin)
[2] Malkin I A and Man'ko V I 1979 *Dynamical Symmetries and Coherent States of Quantum Systems* (Moscow: Nauka) p 320
[3] Klauder I R and Skagerstam B S 1985 *Coherent States, Applications in Physics and Mathematical Physics* (Singapore: World Scientific) p 991
[4] Perelomov A M 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[5] Bagrov V G and Gitman D M 1990 *Exact Solutions of Relativistic Wave Equations* (Dordrecht: Kluwer) p 323
[6] Perelomov A M 1972 *Commun. Math. Phys.* **26** 222
[7] Rasetti M 1975 *Int. J. Theor. Phys.* **13** 425
[8] Berezin F A 1975 *Commun. Math. Phys.* **40** 153
[9] Berezin F A 1986 *The Method of Secondary Quantization* (Moscow: Nauka)
[10] Bloch F 1946 *Phys. Rev.* **70** 460
Bloch F, Hansen W W and Packard M 1946 *Phys. Rev.* **70** 474
[11] Radcliff J M 1971 *J. Phys. A: Math. Gen.* **4** 313

[12] Arecchi F T, Courtens E, Gilmor R and Thomas H 1972 *Phys. Rev.* 6 2211
[13] Lieb E M 1973 *Commun. Math. Phys.* **31** 327
[14] Bellisard J and Holts R 1974 *J. Math. Phys.* **15** 1275
[15] Delborgo R and Fox J R 1977 *J. Phys. A: Math. Gen.* **10** 223
Delbourgo R 1977 *J. Phys. A: Math. Gen.* **10** 1837
[16] Simon B 1980 *Commun. Math. Phys.* **71** 247
[17] Yaffe L G 1982 *Rev. Mod. Phys.* **54** 407
[18] Korchemsky G P 1992 *Int. J. Mod. Phys.* A **7** 339
[19] Katriel J, Rasetti M and Solomon A I 1987 *Phys. Rev.* D **35** 2601
[20] Gilmor R 1972 *Ann. Phys., NY* **74** 391
[21] Gitman D M and Shelepin A L 1990 *Proc. XVIII Int. Coll. on Group Theoretical Methods in Physics (Moscow, June 1990)* (New York: Nova Science) p 251
Gitman D M, Kharchev S M and Shelepin A L 1990 *Trudy PhIAN (Proc. Lebedev Inst.)* **201** 95
Gitman D M and Shelepin A L 1991 *Preprint* MIT, CTP 1990, p 32
[22] Kobayashi S and Nomizu K 1969 *Foundation of Differential Geometry* vol 2 (New York: Interscience)
[23] Schwinger J 1965 *Quantum Theory of Angular Momentum* ed L C Biedenharn and H van Dam (New York: Academic)
[24] Raghunathan K, Seetharaman M and Vasan S S 1989 *J. Phys. A: Math. Gen.* **22** L1089
[25] Berezin F A 1980 *Usp. Fiz. Nauk* **132** 497
Berezin F A and Shubin M A 1983 *Schrödinger Equation* (Moscow: Moscow State University)